Generalizations of infinite iterated function systems

ASIST. DR. DAN DUMITRU

UNIVERSITATEA "SPIRU HARET" DIN BUCUREŞTI

DEPARTAMENTUL DE MATEMATICĂ ŞI INFORMATICĂ

CAMAI 12-13 IUNIE 2015

- In this paper we study generalized infinite iterated funcion systems (GIIFS) formed by Meir-Keeler functions.
- **Definition 1:** Let (X, d) be a metric space. A function $f: X \to X$ is called a *Meir-Keeler function* if for every $\eta > 0$, there exists $\delta > 0$ such that $d(x, y) < \eta + \delta$ implies $d\big(f(x), f(y)\big) < \eta$.
- Usually, when we deal with functions of multiple variables we have three types of Meir-Keeler functions. Thus:
- **Definition 2:** Let (X, d) be a metric space and $m \in \mathbb{N}^*$. A function $f: X^m \to X$ is said to be:

- *weak Meir-Keeler* if for every $\eta > 0$, there exists $\delta > 0$ such that $\eta \leq d(x_i, y_i) < \eta + \delta$ for every $i = \overline{1, m}$ implies

$$d\big(f(x_1, \dots, x_m), f(y_1, \dots, y_m)\big) < \eta.$$

- *Meir-Keeler* if for every $\eta_i > 0$, there exists $\delta > 0$ such that $\eta_i \le d(x_i, y_i) < \eta_i + \delta$ for every $i = \overline{1, m}$ implies

$$d\big(f(x_1, \dots, x_m), f(y_1, \dots, y_m)\big) < \max_{i=\overline{1,m}} \eta_i.$$

- *strong Meir-Keeler* if for every $\eta > 0$, there exists $\delta > 0$ such that $d(x_i, y_i) < \eta + \delta$ for every $i = \overline{1, m}$ implies

$$d\big(f(x_1, \dots, x_m), f(y_1, \dots, y_m)\big) < \eta.$$

- **<u>Definition 3:</u>** Let (X, d) be a metric space and $m \in \mathbb{N}^*$. A family of functions $f_k: X^m \to X$, $k \in I$ is called *uniformly strong Meir-Keeler* if for every $\eta > 0$, there exist $\delta, \lambda > 0$ such that $d(x_i, y_i) < \eta + \delta$ for every $i = \overline{1, m}$ implies $d(f_k(x_1, \dots, x_m), f_k(y_1, \dots, y_m)) \le \eta - \lambda$ for every $k \in I$.
- We remark that Meir-Keeler implies weak Meir-Keeler, strong Meir-Keeler implies weak Meir-Keeler and a Meir-Keeler function has the Lipschitz constant less or equal to 1.

- **Definition 4:** Let (X, d) a complete metric space and $f_k: X^m \to X$ a continuous function for every $k \in I$, where $m \in \mathbb{N}^*$.

- $(f_k)_{k\in I}$ is called *bounded* if for every bounded sets $B_1, \ldots, B_m \subset X$ we have $\bigcup_{k\in I} f_k(B_1, \ldots, B_m)$ bounded.

- A *generalized infinite iterated function system* (GIIFS) consists of a bounded family of continuous functions $f_k: X^m \to X$ for every $k \in I$, where $m \in \mathbb{N}^*$ and it is denoted by $S^m = (X^m, (f_k)_{k\in I})$.

- The *fractal operator* associated to a (GIIFS) is defined by $F_{S^m}: B(X)^m \to B(X)$, $F_{S^m}(B_1, \ldots, B_m) = \overline{\bigcup_{k\in I} f_k(B_1, \ldots, B_m)}$, where $B(X)$ is the set of nonempty bounded closed subsets of X.

- **Theorem 5:** *Let (X, d) be a complete metric space and $S^m = (X^m, (f_k)_{k\in I})$ a (GIIFS). If the family $(f_k)_{k\in I}$ is uniformly strong Meir-Keeler, then the fractal operator F_{S^m} is strong Meir-Keeler (in particular F_{S^m} is weak Meir-Keeler).*

- **Theorem 6:** *Let (X, d) be a complete metric space and $f: X^m \to X$ a strong Meir-Keeler function. Then there exists a unique point $a \in X$ such that $f(a, \dots, a) = a$ (i.e. f has a unique fixed point).*
- From theorems 5 and 6 we can conclude the following:
- **Theorem 7:** *Let (X, d) be a complete metric space and $S^m = (X^m, (f_k)_{k \in I})$ a (GIIFS) such that the family $(f_k)_{k \in I}$ is uniformly strong Meir-Keeler. Then there exists a unique set $A \in B(X)$ such $F_{S^m}(A, \dots, A) = A$ (A is called the attractor of S^m).*
- **Example 8:** Let $X = [0,1]$ and $f_n: [0,1]^2 \to [0,1], f_n(x, y) = \frac{x+y}{2^{n+2}} + \frac{1}{2^{n+1}}$ for every $x, y \in [0,1]$ and $n \in \mathbb{N}$. Then from straight computation one can prove that for every $\eta > 0$, there exist $\delta = \lambda = \frac{\eta}{3} > 0$ such that $|x_1 - x_2| < \eta + \delta$ and $|y_1 - y_2| < \eta + \delta$ implies

$$|f_n(x_1, y_1) - f_n(x_2, y_2)| \leq \eta - \lambda.$$

Thus the family $(f_n)_{n\geq 0}$ is uniformly strong Meir-Keeler and hence the fractal operator F_{S^2} is strong Meir-Keeler. In this way, we have proved that there exists a unique set $A \in B([0,1])$ such that $F_{S^2}(A, A) = A$. We remark that $A = [0,1]$.

- Similar results can be obtained for φ – functions.
- **Question 9:** In the conditions of theorem 5, can we have F_{S^m} Meir-Keeler?